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Maximal Points of Stable and Related Polynomials

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We prove that any polynomial having all its roots in a closed half-plane, whose boundary contains the origin, has either one or two maximal points, and only one if it has at least one root in the open half-plane. This result concerns stable polynomials as well as polynomials having only real roots, including real orthogonal polynomials.

Key Words: maximal points; stable polynomials; real roots; orthogonal polynomials.

Let us first recall that a *maximal point* of a polynomial P is any complex number z_0 of modulus 1 such that

$$|P(z_0)| = \|P\|_\infty = \max_{|z|=1} |P(z)|.$$

The first elementary result about maximal points of polynomials asserts that any polynomial of degree n (except for monomials) has at most n maximal points. To see this, it suffices to notice that the function $f(t) = |P(e^{it})|^2$ is a trigonometric polynomial of degree at most n and thus has at most n local maxima on $[0, 2\pi)$.

The next natural question arising about maximal points concerns their proximity to the roots. The problem was raised by Turán (see [3]). Using the well-known Bernstein inequality on the supremum norm of the derivative of a polynomial over the unit circle, that is:

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|$$

for any polynomial P of degree n, it is clear that no maximal point can be "too close" to a root of the polynomial. More precisely, Turán proved that

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if z_0 is a maximal point and α a root with modulus 1 of a polynomial of degree n then

$$|z_0 - \alpha| \geqslant 2 \sin\left(\frac{\pi}{2n}\right).$$

Without any restriction on the modulus of α, Hylten-Cavallius (see [2]) proved that

$$|z_0 - \alpha| \geqslant \frac{2 \sin \dfrac{\pi}{2n}}{1 + \sin \dfrac{\pi}{2n}}.$$

Apart from these results on the proximity to the roots, not much is known about the number and position of the maximal points of a polynomial in relation to the roots. In [1] we were led to study the maximal points of polynomials all of whose roots are of modulus 1. We observed and made use of the following result:

PROPOSITION 1. *Let* $P(z) = \lambda \prod_{j=1}^{n} (z - e^{i\varphi_j})$, *with* $-\pi \leqslant \varphi_1 \leqslant \cdots \leqslant \varphi_n \leqslant \pi$, *be a polynomial with all roots of modulus* 1. *Then on each arc* $(\varphi_j, \varphi_{j+1})$ *of the unit circle between two consecutive roots*, *P has at most one maximal point.*

Proof. Using again the fact that the function $f(t) = |P(e^{it})|^2$ is a trigonometric polynomial of degree n and thus has at most n local maxima on $[0, 2\pi)$, the result is immediate if the polynomial P has n distinct roots. Now suppose that the conclusion failed for a polynomial P with some multiple roots. Then one could slightly perturb P such that the resulting polynomial has n distinct roots of modulus 1, but that the conclusion of the proposition fails to hold. This is the desired contradiction. ∎

This result is of special interest in the case where the polynomial P has only roots of modulus 1 and in a closed half-plane H whose boundary contains the origin, with at least one root in the interior of H. Indeed, in this case it is easy (see the proof of Theorem 2 below) to see that the polynomial P has all its maximal points in the complementary half-plane, and thus P has only one maximal point by Proposition 1. Of course, if the polynomial has all of its roots on the boundary ∂H of H (and still of modulus 1), then by the same argument it has either only one maximal point, which is then on ∂H, or two maximal points, which are then symmetrically located with respect to ∂H.

One might then wonder whether this uniqueness result remains valid without the condition on the moduli of the roots. The answer is affirmative and this is our main theorem.

THEOREM 2. *Let P be a polynomial (but not a monomial) all of whose roots lie in a closed half-plane H whose boundary ∂H contains the origin.*

(1) *If P has at least one root in the interior of H, then P has exactly one maximal point z_0 which is located in the interior of the complementary half-plane.*

(2) *If all the roots of P lie on ∂H, then P has either exactly one maximal point located on ∂H, or two maximal points which are symmetrically located with respect to ∂H.*

Proof. Rotating the roots, we may assume that the roots of P lie in $H = \{\Re(z) \leqslant 0\}$.

Writing

$$P(z) = \lambda \prod_{j=1}^{n} (z - \rho_j e^{i\varphi_j}), \qquad \text{with} \quad \frac{\pi}{2} \leqslant \varphi_j \leqslant \frac{3\pi}{2},$$

we consider the real function

$$f(t) = |P(e^{it})|^2 = |\lambda|^2 \prod_{j=1}^{n} (1 + \rho_j^2 - 2\rho_j \cos(t - \varphi_j))$$

and its logarithmic derivative

$$g(t) = \frac{f'(t)}{f(t)} = 2 \sum_{j=1}^{n} \frac{\rho_j \sin(t - \varphi_j)}{1 + \rho_j^2 - 2\rho_j \cos(t - \varphi_j)}.$$

(1) Suppose that P has at least one root in the open half-plane $\{\Re(z) < 0\}$.

Here we first prove that every maximal point z_0 of P satisfies $\Re(z_0) > 0$. For, we first observe that $|P(e^{it})| > |P(e^{i(\pi - t)})|$ for every t, $|t| < \pi/2$, which ensures that $\Re(z_0) \geqslant 0$ for every maximal point z_0. Thus, we only have to make sure that neither i nor $-i$ is a maximal point of P. Of course, if $P(i) = 0$, then i is not a maximal point for P. Now, if $P(i) \neq 0$, then we have

$$g\left(\frac{\pi}{2}\right) = 2 \sum_{j=1}^{n} \frac{\rho_j \cos(\varphi_j)}{1 + \rho_j^2 - 2\rho_j \sin(\varphi_j)} < 0,$$

so $f'(\pi/2) < 0$ and $\pi/2$ is not a local maximum for f. Therefore i is not a maximal point for P and neither is $-i$ for the same reason.

Then, to prove our claim, we have to show that P has only one maximal point in the open right half-plane $\{\Re(z)>0\}$. In fact we prove a stronger result, namely that f has only one local maximum in $(-\pi/2, \pi/2)$. For this purpose it suffices to show that f has no local minimum in this interval.

Note that if all the roots of P are of modulus 1, then we get

$$g(t)=\sum_{j=1}^{n} \cot\left(\frac{t-\varphi_j}{2}\right)$$

for $t \neq \varphi_j$, $1 \leqslant j \leqslant n$, and

$$g'(t)=-\frac{1}{2}\sum_{j=1}^{n} \frac{1}{\left(\sin\left(\frac{t-\varphi_j}{2}\right)\right)^2}<0,$$

so that g and thus f' vanishes exactly once in $(-\pi/2, \pi/2)$. In the general case, this inequality is not true any more and we have to use a more careful argument.

Let t_0, $|t_0|<\pi/2$, be a zero of f'. We will prove that $f''(t_0)<0$. Writing

$$g'(t_0)=\frac{f''(t_0)\, f(t_0)-(f'(t_0))^2}{(f(t_0))^2}=\frac{f''(t_0)}{f(t_0)},$$

we have to show that $g'(t_0)<0$.

But

$$g'(t_0)=2\sum_{j=1}^{n} \frac{\rho_j \cos(t_0-\varphi_j)}{1+\rho_j^2-2\rho_j\cos(t_0-\varphi_j)}-4\sum_{j=1}^{n}\left(\frac{\rho_j \sin(t_0-\varphi_j)}{1+\rho_j^2-2\rho_j\cos(t_0-\varphi_j)}\right)^2,$$

so

$$g'(t_0)\leqslant 2\sum_{j=1}^{n} \frac{\rho_j \cos(t_0-\varphi_j)}{1+\rho_j^2-2\rho_j\cos(t_0-\varphi_j)}$$

and it suffices to prove that

$$A=\sum_{j=1}^{n} \frac{\rho_j \cos(t_0-\varphi_j)}{1+\rho_j^2-2\rho_j\cos(t_0-\varphi_j)}<0.$$

With this aim, we write, since $g(t_0) = 0$,

$$
\begin{aligned}
A &= A + i\frac{g(t_0)}{2} \\
&= \sum_{j=1}^{n} \frac{\rho_j e^{i(t_0 - \varphi_j)}}{1 + \rho_j^2 - 2\rho_j \cos(t_0 - \varphi_j)} \\
&= e^{it_0} \sum_{j=1}^{n} \frac{\rho_j e^{i(-\varphi_j)}}{1 + \rho_j^2 - 2\rho_j \cos(t_0 - \varphi_j)}.
\end{aligned}
$$

Therefore the real number A can be obtained by rotating the complex number

$$
\sum_{j=1}^{n} \frac{\rho_j e^{i(-\varphi_j)}}{1 + \rho_j^2 - 2\rho_j \cos(t_0 - \varphi_j)}
$$

through an angle of t_0. By hypothesis, this complex number is located in the open half-plane $\{\Re(z) < 0\}$. Since $|t_0| < \pi/2$, this ensures that $A < 0$ and completes the proof in case (1).

(2) Suppose that all the roots of P are purely imaginary.

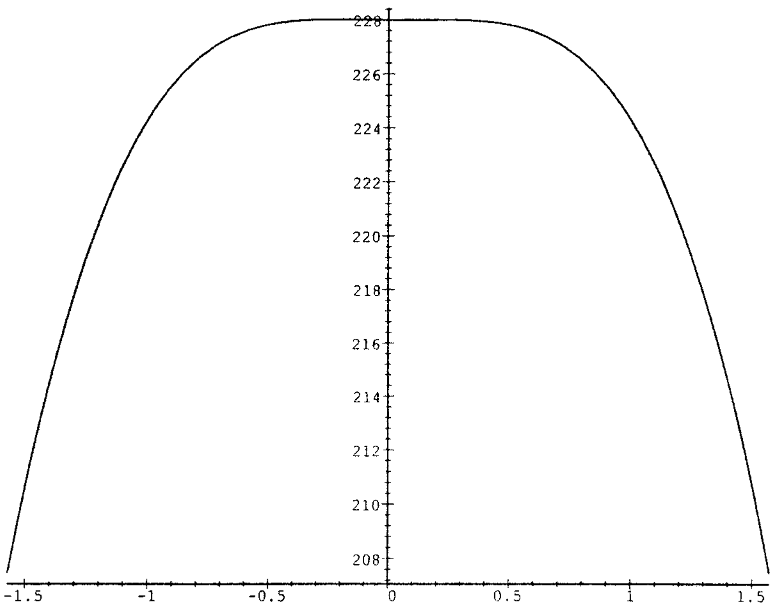

Fig. 1. The function $f(t) = |P(e^{it})|$ on $(-\pi/2, \pi/2)$.

Then, since $|P(e^{it})| = |P(e^{i(\pi - t)})|$ for every t, the maximal points of P are symmetric with respect to the imaginary axis. Thus, to prove our claim we have to show that P has only one maximal point in the closed right half-plane. In fact, it is easy to deduce from the previous case that here again, f has no local minimum in $(-\pi/2, \pi/2)$. Indeed, if f had such a local minimum, then one could perturb P by moving a root into the open left half-plane. But this would contradict the result of case (1). ∎

Remark 1. Two important classes of polynomials are involved in this result: stable polynomials; that is, polynomials with all roots with negative real parts (arising in the study of dynamical systems and their stability), and totally real polynomials; that is, real polynomials having only real roots (including real orthogonal polynomials).

Remark 2. Looking at the proof of Theorem 2, one can see that we actually prove a stronger result, namely that the derivative f' of the real function $f(t) = |P(e^{it})|$ has at most one zero in $(-\pi/2, \pi/2)$ (although the function f is not necessarily convex on this interval as it was in Proposition 1). Of course, this zero gives a (or *the*) maximal point of the polynomial, which allows quick and reliable algorithms to compute the L_∞-norm of such polynomials since only one zero of the trigonometric polynomial f' is to be calculated.

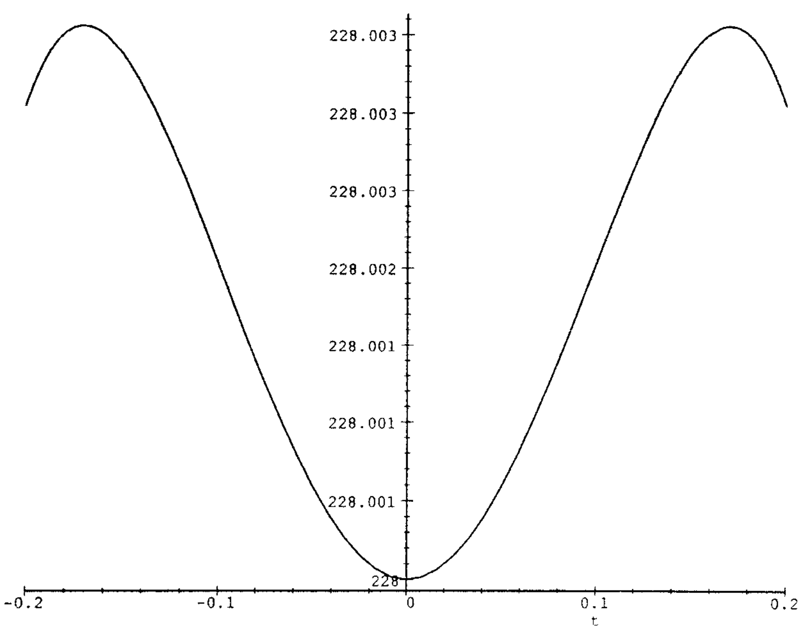

Fig. 2. The function $f(t) = |P(e^{it})|$ on $(-0.2, 0.2)$.

Of course, one might wonder whether Proposition 1 can be extended to a result for unrestricted polynomials, which would imply Theorem 2, namely:

Is it true that for any polynomial (not a monomial) $P(z) = \lambda \prod_{j=1}^{n} (z - p_j e^{i\varphi_j})$, *with* $-\pi \leqslant \varphi_1 \leqslant \cdots \leqslant \varphi_n \leqslant \pi$, *each arc* $(\varphi_j, \varphi_{j+1})$ *of the unit circle between two consecutive arguments of roots contains at most one maximal point of* P?

The answer is negative as shown by the following counter-example. Let P be the polynomial

$$P(z) = (z+1)(z-9)^2 + 100 = z^3 - 17z^2 + 63z + 181.$$

Then the arguments of the roots of P are π, -0.309 and 0.309, but the associated function $f(t) = |P(e^{it})|$ has local maxima at -0.169 and 0.169 (see Figs. 1 and 2).

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